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# Chaotic continua of continuum-wise expansive homeomorphisms

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## 1 Introduction.

In this note, we consider the following problem:

**Problem 1.1.** *If  $f : X \rightarrow X$  is an expansive (or a continuum-wise expansive) homeomorphism of a one-dimensional continuum  $X$ , does  $X$  contain an indecomposable subcontinuum? Moreover, what kinds of dynamical structures does such an indecomposable continuum admit? Is each chaotic continuum of  $f$  indecomposable?*

In this note, we will give some partial answers in the affirmative to the above problem. It is well known that every continuum  $X$  with  $\dim X \geq 2$  contains an indecomposable subcontinuum. Also there is an expansive homeomorphism  $f$  on the 2-dimensional torus  $T^2$  such that  $T^2$  is the only chaotic continuum of  $f$  and hence  $T^2$  is the decomposable chaotic continuum of  $f$ . In [5] and [6], we investigated chaotic continuum of homeomorphism. We proved the existence of (minimal) chaotic continuum of continuum-wise expansive homeomorphism and we also investigated the indecomposability of chaotic continua and their composants. In fact, we proved that if  $G$  is a finite graph and  $f : X \rightarrow X$  of a  $G$ -like continuum  $X$  is a continuum-wise expansive homeomorphism, then there is an indecomposable chaotic continuum of  $f$ . In [9], Mouron proved the existence of an indecomposable subcontinuum of  $X$  for the case that  $X$  is a  $k$ -cyclic continuum ( $k < \infty$ ) and  $X$  admits an expansive homeomorphism. In this note, we define the notion of *closed subset having uncountable handles* and we show that if  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a continuum  $X$  and  $Z$  is a minimal chaotic continuum of  $f$ , then for each proper closed subset  $A$  of  $Z$  with  $\text{Int}_Z A \neq \emptyset$ ,  $A$  has uncountable handles in  $Z$ . As a corollary, we see that if  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism and  $X$  does not contain any subcontinuum having uncountable handles, then each minimal chaotic continuum of  $f$  is indecomposable. This implies a stronger result than the Mouron's theorem above [9]. In fact, we obtain that if  $X$  is a  $k$ -cyclic continuum and  $X$  admits a continuum-wise expansive homeomorphism  $f$ , then each minimal chaotic continuum of  $f$  is indecomposable. The proof is different from the methods of the proof of Mouron [9].

## 2 Expansive homeomorphism and continuum-wise expansive homeomorphism.

All spaces considered in this note are assumed to be separable metric spaces. By a *compactum* we mean a compact metric space. A *continuum* is connected, nondegenerate compactum. A homeomorphism  $f : X \rightarrow X$  of a compactum  $X$  with metric  $d$  is called *expansive* ([15]) if there is  $c > 0$  such that for any  $x, y \in X$  and  $x \neq y$ , then there is an integer  $n \in \mathbb{Z}$  such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism  $f : X \rightarrow X$  of a compactum  $X$  is *continuum-wise expansive* (resp. *positively continuum-wise expansive*) [4] if there is  $c > 0$  such that if  $A$  is a nondegenerate subcontinuum of  $X$ , then there is an integer  $n \in \mathbb{Z}$  (resp. a positive integer  $n \in \mathbb{N}$ ) such that

$$\text{diam } f^n(A) > c,$$

where  $\text{diam } B = \sup\{d(x, y) \mid x, y \in B\}$  for a set  $B$ . Such a positive number  $c$  is called an *expansive constant* for  $f$ . Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (see [4]). These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [1]-[3], [8], [12]-[15]).

The hyperspace  $2^X$  of  $X$  is the set all nonempty closed subsets of  $X$  with the *Hausdorff metric*  $d_H$ . Let

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\}.$$

Note that  $2^X$  and  $C(X)$  are compact metric spaces (e.g., see [7] or [11]). For a homeomorphism  $f : X \rightarrow X$ , we define sets of stable and unstable nondegenerate subcontinua of  $X$  as follows (see [6]):

$$\mathbf{V}^s(= \mathbf{V}_f^s) = \{A \mid A \text{ is a nondegenerate subcontinuum of } X \text{ such that} \\ \lim_{n \rightarrow \infty} \text{diam } f^n(A) = 0\},$$

$$\mathbf{V}^u(= \mathbf{V}_f^u) = \{A \mid A \text{ is a nondegenerate subcontinuum of } X \text{ such that} \\ \lim_{n \rightarrow \infty} \text{diam } f^{-n}(A) = 0\}.$$

For each  $0 < \delta < \epsilon$ , put

$$\mathbf{V}^s(\delta; \epsilon) = \{A \in C(X) \mid \text{diam } A \geq \delta, \text{ and } \text{diam } f^n(A) \leq \epsilon \text{ for each } n \geq 0\} \\ \mathbf{V}^u(\delta; \epsilon) = \{A \in C(X) \mid \text{diam } A \geq \delta, \text{ and } \text{diam } f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0\}.$$

Similarly, for each closed subset  $Z$  of  $X$  and  $x \in Z$ , the *continuum-wise  $\sigma$ -stable sets*  $V^\sigma(x; Z)$  ( $\sigma = s, u$ ) of  $f$  are defined as follows:

$$V^s(x; Z) = \{y \in Z \mid \text{there is } A \in C(Z) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^n(A) = 0\}, \\ V^u(x; Z) = \{y \in Z \mid \text{there is } A \in C(Z) \text{ such that } x, y \in A \text{ and } \lim_{n \rightarrow \infty} \text{diam } f^{-n}(A) = 0\}.$$

A subcontinuum  $Z$  of  $X$  is called a  *$\sigma$ -chaotic continuum* of  $f$  (where  $\sigma = s, u$ ) if

1. for each  $x \in Z$ ,  $V^\sigma(x; Z)$  is dense in  $Z$ , and
2. there is  $\tau > 0$  such that for each  $x \in Z$  and each neighborhood  $U$  of  $x$  in  $X$ , there is  $y \in U \cap Z$  such that

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) \geq \tau \text{ in case } \sigma = s, \text{ or}$$

$$\liminf_{n \rightarrow \infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau \text{ in case } \sigma = u.$$

A subcontinuum  $Z$  of  $X$  is called a *minimal  $\sigma$ -chaotic continuum* of  $f$  (where  $\sigma = s, u$ ) if  $Z$  is a  $\sigma$ -chaotic continuum of  $f$  and  $Z$  does not contain any proper  $\sigma$ -chaotic continuum of  $f$ . In this note, we often abbreviate  $\sigma$ -chaotic continuum to chaotic continuum. Note that  $V^\sigma(\delta; \epsilon)$  ( $\sigma = u, s$ ) is closed in  $C(X)$ . Also, note that if  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism with an expansive constant  $c > 0$ , then (1) for each  $0 < \delta < \epsilon < c$ ,  $V^\sigma(\delta; \epsilon) \subset V^\sigma$ , and  $V^\sigma$  is an  $F_\sigma$ -set in  $C(X)$ , and (2)  $V^u(z; Z)$  is a connected  $F_\sigma$ -set containing  $z$ , because

$$V^u(z; Z) = \bigcup_{n=0}^{\infty} (\bigcup \{A \in C(Z) \mid z \in A, \text{diam } f^{-i}(A) \leq \epsilon \text{ for } i \geq n\}) \text{ (see [4, (2.1)]).}$$

Similarly,  $V^s(z; Z)$  is a connected  $F_\sigma$ -set containing  $z$ . In [5], we showed that if  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a compactum  $X$  with  $\dim X > 0$ , then there exists a minimal chaotic continuum of  $f$  (see [5, (3.6)]). In this case, if  $Z$  is a  $\sigma$ -chaotic continuum of  $f$ , then the decomposition  $\{V^\sigma(z; Z) \mid z \in Z\}$  of  $Z$  is an uncountable family of mutually disjoint, dense connected  $F_\sigma$ -sets in  $Z$ .

A continuum  $X$  is *decomposable* if there are two proper subcontinua  $A$  and  $B$  of  $X$  such that  $A \cup B = X$ . A continuum  $X$  is *indecomposable* if it is not decomposable. Let  $X$  be a continuum and let  $p \in X$ . Then the set

$$c(p) = \{x \in X \mid \text{there is a proper subcontinuum } A \text{ of } X \text{ containing } p \text{ and } x\}$$

is called the *composant* of  $X$  containing  $p$ . Note that if  $X$  is an indecomposable continuum, then  $\{c(p) \mid p \in X\}$  is an uncountable family of mutually disjoint, dense connected  $F_\sigma$ -sets in  $X$ . See [7] for some fundamental properties of indecomposable continua and composants. A closed subset  $A$  of  $X$  has *uncountable handles* if there is a family  $\{H_\alpha \mid \alpha \in \Lambda\}$  of mutually disjoint nondegenerate subcontinua  $H_\alpha$  (i.e.,  $H_\alpha \cap H_\beta = \emptyset$  for  $\alpha \neq \beta$ ) of  $X$  such that each  $A \cap H_\alpha (\neq \emptyset)$  has at least two components and  $\Lambda$  is an uncountable set. A continuum  $X$  is *k-cyclic* if for any  $\epsilon > 0$ , there is a finite open cover  $\mathcal{U}$  of  $X$  such that  $\text{mesh}(\mathcal{U}) < \epsilon$  and the nerve  $N(\mathcal{U})$  of  $\mathcal{U}$  is a one-dimensional polyhedron which has at most  $k$  distinct simple closed curves.

### 3 Results.

**Proposition 3.1.** *If a continuum  $X$  is  $k$ -cyclic for some  $k < \infty$ , then  $X$  contains no subcontinuum having uncountable handles.*

**Remark.** The converse assertion of the above proposition is not true. Hawaiian earring  $H$  contains no subcontinuum having uncountable handles and  $H$  is not  $k$ -cyclic for any  $k < \infty$ .

**Lemma 3.2.** (see [5, (3.2)]) *Let  $f : X \rightarrow X$  be a continuum-wise expansive homeomorphism of a compactum  $X$  with an expansive constant  $c > 0$ , and let  $0 < \epsilon < c/2$ . Then there is  $\delta > 0$  such that if  $A$  is a subcontinuum of  $X$  with  $\text{diam } A \leq \delta$  and  $\text{diam } f^m(A) \geq \epsilon$  for some  $m \in \mathbb{Z}$ , then one of the following two conditions holds:*

1. If  $m \geq 0$ , for each  $n \geq m$  and  $x \in f^n(A)$ , there is a subcontinuum  $B$  of  $A$  such that  $x \in f^n(B)$ ,  $\text{diam } f^j(B) \leq \epsilon$  for  $0 \leq j \leq n$  and  $\text{diam } f^n(B) = \delta$ .
2. If  $m < 0$ , for each  $n \geq -m$  and  $x \in f^{-n}(A)$ , there is a subcontinuum  $B$  of  $A$  such that  $x \in f^{-n}(B)$ ,  $\text{diam } f^{-j}(B) \leq \epsilon$  for  $0 \leq j \leq n$ , and  $\text{diam } f^{-n}(B) = \delta$ .

**Lemma 3.3.** ([5, (3.3) and (3.4)]) *Let  $f : X \rightarrow X$  be a continuum-wise expansive homeomorphism of a compactum  $X$  with  $\dim X > 0$ . Then the following are true.*

1.  $V^u \neq \emptyset$  or  $V^s \neq \emptyset$ .
2. If  $\delta > 0$  is as in the above lemma, then for each  $\gamma > 0$  there is a natural number  $N(\gamma)$  such that if  $A$  is a subcontinuum of  $X$  with  $\text{diam } A \geq \gamma$ , then either  $\text{diam } f^n(A) \geq \delta$  for each  $n \geq N(\gamma)$  or  $\text{diam } f^{-n}(A) \geq \delta$  for each  $n \geq N(\gamma)$  holds.

**Theorem 3.4.** *If  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a continuum  $X$  and  $Z$  is a minimal chaotic continuum of  $f$ , then for any proper closed subset  $A$  of  $Z$  with  $\text{Int}_Z A \neq \emptyset$ ,  $A$  has uncountable handles. Moreover,  $Z$  is decomposable if and only if there exists a proper subcontinuum  $C$  of  $Z$  with  $\text{Int}_Z C \neq \emptyset$  such that  $C$  has uncountable handles in  $Z$ .*

**Corollary 3.5.** *Suppose that a continuum  $X$  contains no subcontinuum having uncountable handles. If  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of  $X$  and  $Z$  is a minimal chaotic continuum of  $f$ , then  $Z$  is indecomposable.*

**Corollary 3.6.** *If  $X$  is a  $k$ -cyclic continuum for some  $k < \infty$  and  $X$  admits a continuum-wise expansive homeomorphism  $f$ , then each minimal chaotic continuum of  $f$  is indecomposable.*

Next, we consider the following problem.

**Problem 3.7.** *Suppose that  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a one-dimensional continuum  $X$  and  $Z$  is an indecomposable  $\sigma$ -chaotic continuum of  $f$ . Does the composant  $c(z)$  of  $Z$  containing  $z$  coincide with  $V^\sigma(z; Z)$  for each  $z \in Z$ ?*

We give a partial answer in the affirmative to the problem. A subcontinuum  $A$  of  $X$  has *uncountable handlebars* if there is a family  $\{H_\alpha | \alpha \in \Lambda\}$  of mutually disjoint nondegenerate subcontinua  $H_\alpha$  of  $X$  such that  $H_\alpha - A \neq \emptyset$ ,  $A \cap H_\alpha \neq \emptyset$  for each  $\alpha \in \Lambda$  and  $\Lambda$  is an uncountable set. Note that if a subcontinuum  $A$  of  $X$  has uncountable handles, then  $A$  has uncountable handlebars. A continuum  $X$  is  *$k$ -branched* ( $k \in \mathbb{N}$ ) if for any  $\epsilon > 0$ , there is a finite open cover  $\mathcal{U}$  of  $X$  such that  $\text{mesh}(\mathcal{U}) < \epsilon$  and the nerve  $N(\mathcal{U})$  is a one-dimensional polyhedron which has at most  $k$  distinct branch points. Note that Hawaiian earring  $H$  is a 1-branched continuum.

**Proposition 3.8.** *If a continuum  $X$  is  $k$ -branched for some  $k < \infty$ , then  $X$  contains no subcontinuum having uncountable handlebars.*

We need the following lemma.

**Lemma 3.9.** (Sum theorem of dimension) *If  $X_i$  ( $i \in \mathbb{N}$ ) are closed subsets of a separable metric space  $X$  such that  $\dim X_i \leq n$  and  $X = \bigcup_{i \in \mathbb{N}} X_i$ , then  $\dim X \leq n$ .*

**Theorem 3.10.** *Suppose that a continuum  $X$  contains no subcontinuum having uncountable handlebars. If  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism, then there is a  $\sigma$ -chaotic continuum  $Z$  of  $f$  such that  $Z$  is an indecomposable continuum and for each  $z \in Z$ , the composant  $c(z)$  of  $Z$  containing  $z$  coincides with  $V^\sigma(z; Z)$ .*

**Corollary 3.11.** *If  $f : X \rightarrow X$  is a continuum-wise expansive homeomorphism of a  $k$ -branched continuum  $X$  ( $k < \infty$ ), then there is a  $\sigma$ -chaotic continuum  $Z$  of  $f$  ( $\sigma = s$  or  $u$ ) such that  $Z$  is an indecomposable continuum such that for each  $z \in Z$ , the composant  $c(z)$  of  $Z$  containing  $z$  coincides with  $V^\sigma(z; Z)$ .*

In [10], MOURON proved that if  $f : X \rightarrow X$  is an expansive homeomorphism, then  $X$  is not tree-like. We will give a more precise result than Mouron's result. We need the following simple lemmas.

**Lemma 3.12.** *Let  $(X, d)$  be a metric space and let  $\delta > 0$ . Then for each positive integer  $n$ , there is a positive number  $\eta = \eta(\delta, n) > 0$  such that if  $A$  is any connected subset  $M$  of  $X$  with  $\text{diam}(M) \geq \delta$ , then there are distinct points  $y_i$  ( $i = 1, 2, \dots, n$ ) in  $M$  such that  $d(y_i, y_j) \geq \eta$  for  $i \neq j$ .*

**Lemma 3.13.** *Let  $f : X \rightarrow X$  be an expansive homeomorphism of a compactum  $X$  with an expansive constant  $c > 0$ . For each  $\eta > 0$ , there is a positive integer  $n = n(\eta)$  such that if  $x, y \in X$  with  $d(x, y) \geq \eta$ , then  $\max\{|d(f^i(x), f^i(y))| - n \leq i \leq n\} \geq c$ .*

**Theorem 3.14.** *Let  $f : X \rightarrow X$  be an expansive homeomorphism of a continuum  $X$ . If a subcontinuum  $Y$  of  $X$  satisfies the condition  $P_\sigma(y; Y)$  for some  $y \in Y$ , then  $Y$  is not a tree-like continuum. In particular, every chaotic continuum of  $f$  is not tree-like.*

**Corollary 3.15.** *Suppose that a continuum  $X$  contains no subcontinuum having uncountable handles. If  $f : X \rightarrow X$  is an expansive homeomorphism of  $X$  and  $Z$  is a minimal chaotic continuum of  $f$ , then  $Z$  is an indecomposable continuum which is not tree-like.*

**Corollary 3.16.** *Suppose that a continuum  $X$  contains no subcontinuum having uncountable handlebars. If  $f : X \rightarrow X$  is an expansive homeomorphism, then there is a  $\sigma$ -chaotic continuum  $Z$  of  $f$  such that  $Z$  is not tree-like,  $Z$  is indecomposable and for each  $z \in Z$ , the composant  $c(z)$  of  $Z$  containing  $z$  coincides with  $V^\sigma(z; Z)$ .*

**Remark.** There are many tree-like chaotic continua of continuum-wise expansive homeomorphisms.

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